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Examples of Quantisable Dynamical Systems: The Hydrogen Atom and Automorphism Groups

NORMAN E. HURT†

Department of Mathematics University of Massachusetts, Amhurst, Mass. USA.

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Abstract

The regularised energy surface of the n-dimensional hydrogen atom is shown to be naturally the total space of a quantisable dynamical system. The automorphism groups of dynamical systems are studied; and the connected Riemannian dynamical systems with automorphism groups of maximal dimension are classified. Finally, the compact, connected and simply connected quantisable dynamical system with automorphism group of maximal dimension is shown to be the set of independent harmonic oscillators with equal periods.

Introduction

Recently Onofri & Pauri (1972a, 1972b) have considered the n-dimensional hydrogen atom or the n-dimensional Kepler problem within the framework of dynamical symmetry groups and canonical quantisation. Let $p=(p_1,\ldots,p_n)$ and $q=(q_1,\ldots,q_n)$; then the Hamiltonian is $H=(\|p\|^2/2m)-(k/\|q\|)$. The associated Hamiltonian vector field on the phase space $B^{2n}=T^*(\mathbf{R}^n-\{0\})$ is not complete (in the sense of Kobayashi & Nomizu, 1963, Section I.1); and the surfaces Σ_H of constant negative energy $H=-a^2$ are connected but non-compact. To remove the singularity of $q=0$, a canonical (= symplectic) change of coordinates is performed; then by a compactification of the resulting energy manifold, the Hamiltonian vector field is made complete (by Kobayashi & Nomizu, 1963, Proposition I.1.6). This has been outlined by Moser (1970) (following Levi–Civita (1906) in the case $n=2$) (cf. Onofri & Pauri, 1972b, Section 2.c) and independently by Andrie & Simms (1972) in the case $n=2$ (following Bacry *et al.* (1966), and so Fock (1935) *et alia*); here the new energy surface $\tilde{\Sigma}_H$ is shown to be the unit tangent bundle of the n-dimensional sphere S^n—i.e. the Stiefel manifold $V_{n+1,2}=SO(n+1)/SO(n-1)$ (cf. Steenrod, 1951, Section 7.7). In the case $n=2, \tilde{\Sigma}_H=V_{3,2}=\mathbf{R}P(3)=SO(3)=SO(4)/O(3)$. The first main result of this paper is to show that the energy surface $\tilde{\Sigma}_H$ is

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naturally the total space of a quantisable dynamical system (= QDS). The details on QDSs are to be found in Hurt (1968, 1970a, 1970b, 1970c, 1971a, 1971b, 1972a, 1972b, 1973a, 1973b); cf. also Onofri & Pauri (1972b). We review this subject briefly below.

1. *Quantisable Dynamical Systems and the Hydrogen Atom*

A *dynamical system* (M,Ω) is a $(2n+1)$-dimensional manifold M with a 2-form Ω on M of rank $2n$ (*v.* Hurt, 1971a, Section 2). From Hurt (1971a), Proposition 2.4 and 2.5 a dynamical system (= DS) on M is specified by a triple (ϕ,ω,Z) where ϕ is a tensor field of type $(1,1)$, ω is a 1-form and Z is a vector field on M satisfying the axioms: (1) $\omega(Z)=1$ and (2) $\phi^2 = -Id+\omega\otimes Z$; i.e. (ϕ,ω,Z) is an *almost contact structure* on M. If $d\omega=\Omega$, then (M,Ω) is called a *contact manifold* or a *contact dynamical system* (= cDS).

If the vector field Z for a DS(M,Ω) specified by (ϕ,ω,Z) is proper (= complete), respectively regular in the sense of Palais (1957), then (M,Ω) is said to be a *proper*, respectively *regular*, DS. If the period function (*v.* Hurt, 1971a, Section 5) is a constant, finite or infinite, then the DS(M,Ω) is said to be *finite* or *infinite*. If M is compact, then the DS(M,Ω) is called *compact*; as noted above every vector field is then proper and clearly the DS is then finite. A proper regular finite cDS is called a *quantisable dynamical system* (= QDS) (*v.* Hurt, 1968, 1970a, 1970b, 1970c, 1971a, 1971b, 1972a, 1972b, 1973a, 1973b).

Let G^1 denote a one dimensional Lie Group (compact or non-compact) and let $\mathscr{L}$ denote Lie derivative (*v.* [19]). Then by Hurt (1971a), Proposition 5.3 we have

Proposition 1.1 (Tanno, 1965). If (M,Ω) is a proper regular DS with $\mathscr{L}(Z)\omega=0$, then $G^1\to M\to B$ is a principal G^1-bundle and ω is the connection form; here $G^1=\mathbf{R}$, respectively S^1, as (M,Ω) is infinite, respectively finite.

Corollary 1.2 (Tanno, 1965). If (M,Ω) is a compact, regular cDS (so a QDS), then M is the total space of a principal S^1-bundle over the manifold B.

Due to this principal bundle structure, the manifold M in a DS(M,Ω) is called the *total space* of the DS.

By Oguie (1965) Theorem 1.1 we have

Proposition 1.3. If (M,Ω) is a regular DS with all integral curves of Z homeomorphic and if $\mathscr{L}(Z)\phi=0=\mathscr{L}(Z)\omega$, then $G^1\to M\to B$ is a principal G^1-bundle with connection form ω and a natural almost complex structure J on B induced by (ϕ,ω,Z).

Corollary 1.4. If (M,Ω) is a compact regular DS and $\mathscr{L}(Z)\phi=0=\mathscr{L}(Z)\omega$ then M is a principal S^1-bundle over B.

As noted in Hurt (1971a), Proposition 2.4, every DS admits a Riemannian metric g such that $g(X,Z)=\omega(X)$ and $\Omega(X,Y)=g(X,\phi Y)$. Then (ϕ,ω,Z,g) is called an almost *contact metric* (or *Riemannian*) *structure*. If in this case Z is a Killing vector field (i.e. $\mathscr{L}(Z)g=0$) then (M,Ω) is called a *K-almost contact metric manifold*, or a K-DS. To relate Propositions 1.1 and 1.3 above we quote:

Proposition 1.5 (Tanno, 1965). If (M,Ω) is a proper, regular cDS, then there is an almost contact metric structure (ϕ,ω,Z,g) associated to the contact form ω such that $\mathscr{L}(Z)\phi=0$ (i.e., (M,Ω) is a K-cDS).

Proposition 1.6 (Tanno, 1965). If (M,Ω) is a proper regular DS, then $\mathscr{L}(Z)\phi=0$ iff M is the total space of a principal G^1-bundle with connection form ω and almost complex manifold B as base.

Let $V_{n,k}(\mathbf{R})$ denote the set of orthonormal k-frames in $\mathbf{R}^n$, so $V_{n,k}(\mathbf{R})=O(n,\mathbf{R})/O(n-k,\mathbf{R})=SO(n,\mathbf{R})/SO(n-k,\mathbf{R})=$ the *Stiefel manifolds*. Let $G_{n,k}(\mathbf{R})$, respectively $G'_{n,k}(\mathbf{R})$, denote the set of (respectively oriented) k-planes through the origin. As homogeneous spaces the *Grassmann manifolds* are

$$G_{n,k}(\mathbf{R})=O(n,\mathbf{R})/O(k,\mathbf{R})\times O(n-k,\mathbf{R})$$

respectively

$$G'_{n,k}(\mathbf{R})=SO(n,\mathbf{R})/SO(k,\mathbf{R})\times SO(n-k,\mathbf{R}).$$

Thus

$$O(k,\mathbf{R})\to V_{n,k}(\mathbf{R})\to G_{n,k}(\mathbf{R}) \tag{1.1}$$

and

$$SO(k,\mathbf{R})\to V_{n,k}(\mathbf{R})\to G'_{n,k}(\mathbf{R}) \tag{1.2}$$

are natural fibrations and

$$\mathbf{Z}_2\to G'_{n,k}(\mathbf{R})\to G_{n,k}(\mathbf{R}) \tag{1.3}$$

is the natural two-fold covering.

Let $Q_{n-1}(\mathbf{C})$ denote the *complex quadric* which is defined by the homogeneous equation $\sum_{i=0}^{n} z_i^2=0$ for homogeneous coordinates $\{z_i\}$ of a point in complex projective space $\mathbf{C}P(n)$. Then $Q_{n-1}(\mathbf{C})$ is diffeomorphic to $G'_{n+1,2}(\mathbf{R})$ (*v*. Dieudonne, 1971, XX Section 11). (We note that real quadrics which are QDSs have been studied in [17].)

The main result mentioned above is:

Proposition 1.7. The regularised energy surface $\tilde{\Sigma}_H=V_{n+1,2}(\mathbf{R})$ of the n-dimensional hydrogen atom is naturally the total space of a (normal) QDS.

Proof. First note that (1.2) for case $k=2$ gives

$$S^1\to V_{n+1,2}\to G'_{n+1,2},$$

which is a principal circle bundle over the complex quadric $Q_{n-1}(\mathbf{C})$. By Chern (1969), p. 61, $Q_{n-1}(\mathbf{C})$ is a Hodge manifold; and so by Hurt (1971a), Corollary 6.8 there is a natural normal (or Sasakian) QDS over $Q_{n-1}(\mathbf{C})$, namely (M,Ω) for a natural Ω such that $S^1 \to M \to Q_{n-1}(\mathbf{C})$ is a principal circle bundle. It is easily shown that M is diffeomorphic to $V_{n+1,2}$ (*v.* Kenmotsu, 1970, Theorem 4). QED.

Remark 1. We note that by Boothby & Wang (1958), Corollary, p. 733, if (M,Ω) is a homogeneous QDS (*v.i.*) of dimension $4r+1$ ($r>1$), then M is homeomorphic to the unit tangent bundle U^*X of a manifold X only when $M = V_{2r+2,2}$.

Remark 2. Since $Q_{n-1}(\mathbf{C})$ is an Einstein manifold, then M in Proposition 1.7 above is an ω-Einstein manifold (*v.* Kenmotsu, 1970; Tanno, 1967).

2. *Automorphism Groups and the Harmonic Oscillator*

If $D = (\phi,\omega,Z,g)$ is a contact metric structure on a cDS (M,Ω) and if a certain Nijenhuis tensor field vanishes (*v.* Hurt, 1971a, Section 2), then D is called a *Sasakian* structure and (M,Ω) is called a *normal contact* manifold or a *Sasakian* DS. And by Hurt (1971a), Proposition 2.7 if (M,Ω) is a Sasakian DS, then (M,Ω) is a K-Riemannian cDS.

Let (M,Ω) be a Riemannian cDS specified by $D = (\phi,\omega,Z,g)$. If $\phi^* = \phi$, $Z^* = \alpha^{-1}Z$, $\omega^* = \alpha\omega$ and $g^* = \alpha g + (\alpha^2-\alpha)\omega \otimes \omega$ for a positive constant α, then $D^\alpha = (\phi^*,\omega^*,Z^*,g^*)$ is also a contact metric structure on M, which is called the *D-homothetic deformation.* In fact

Lemma 2.1 (Tanno, 1968). If D is a K-contact (respectively Sasakian) structure on M, then D^α is also a K-contact (respectively Sasakian) structure on M.

If (M,Ω) is a cDS, then the set of diffeomorphisms f of M which satisfy $f^*\omega = \omega$ forms a group, called the group $S(M)$ of *strict contact transformations.* If there is a transitive Lie group G of strict contact transformations on cDS (M,Ω), then (M,Ω) is called a *homogeneous* cDS, or a homogeneous contact manifold in the sense of Boothby & Wang (1958). In this case (M,Ω) is a regular cDS by (Boothby & Wang, 1958), Theorem 4.

If (ϕ,ω,Z) and (ϕ',ω',Z') are two almost contact structures on a manifold M, then the set of diffeomorphisms f of M, which satisfy (1) $f \circ \phi = \phi' \circ f$ and (2) $f(Z) = Z'$, forms a group, $\mathrm{Aut}(M)$. Since

Lemma 2.2 (Morimoto, 1963). If $f \in \mathrm{Aut}(M)$, then $f^*\omega' = \omega$ is true, $\mathrm{Aut}(M)$ is the group of transformations of M which leaves invariant the almost contact structure, or the DS structure, of M; thus $\mathrm{Aut}(M)$ is called the *group of automorphisms* of (M,Ω).

For a suitable topology we have

Proposition 2.3 (Sasaki, 1965–1968). If (M,Ω) is a compact DS, then $\mathrm{Aut}(M)$ is a Lie group.

Proposition 2.4 (Hatakeyama, 1966). If (M,Ω) is a compact regular cDS (hence a QDS), then $S(M)$ acts transitively on M.

Proposition 2.5 (Morimoto, 1963). If (M,Ω) is a compact simply connected homogeneous cDS (hence a QDS), then M has a normal almost contact structure such that $\mathrm{Aut}(M)$ acts transitively on M.

Let $\Phi(M)$ denote the group of all diffeomorphisms of a DS(M,Ω) specified by (ϕ,ω,Z) which leave ϕ invariant.

Proposition 2.6 (Tanno, 1963; Sasaki, 1965–1968, Theorems 25.1, 26.1). If (M,Ω) is a Riemannian cDS, then $\Phi(M)$ is a Lie group, $\dim\Phi(M) \leqslant \dim\mathrm{Aut}(M)+1$, and $\Phi(M)=S(M)$.

Proposition 2.7 (Tanno, 1963; Sasaki, 1965–1968). If (M,Ω) is a compact, Riemannian cDS, then $\Phi(M)=\mathrm{Aut}(M)=S(M)$; and $\Phi(M)\subset\mathrm{Isom}(M)$, where $\mathrm{Isom}(M)$ is the group of isometries of M.

For more details on $\Phi(M)$, $\mathrm{Isom}(M)$, and $\mathrm{Aut}(M)$, cf. Tanno (1963, 1969, 1970).

Proposition 2.8 (Tanno, 1963). If (M,Ω) is a Riemannian cDS and M is an Einstein space, then $\Phi(M)=\mathrm{Aut}(M)$. For example, if (M,Ω) is a K-cDS and M has parallel Ricci tensor (as when (M,Ω) is a K-cDS and M is a symmetric space), then the hypotheses are satisfied.

Proposition 2.9 (Tanno, 1969). If (M,Ω) is a metric DS specified by $D=(\phi,\omega,z,g)$, then the automorphism groups $\mathrm{Aut}(M)$ and $\mathrm{Aut}^*(M)$ with respect to D and D^α, respectively, coincide.

Recall that the sectional curvature of a two-plane (= two-dimensional subspace of the tangent space $T_x(M)$ at point x in M) with orthonormal basis $\{X,Y\}$ is $K(X,Y)=g(R(X,Y)X,Y)$ (*v.* Kobayashi & Nomizu, 1963). Let (ϕ,ω,Z) specify a DS(M,Ω). Then a two-plane is a *ϕ-holomorphic section* if it is spanned by a unit vector X orthogonal to Z (i.e. $\omega(X)=0$) at x and ϕX. Then for the basis $\{X,\phi X\}$ of this two-plane, the *ϕ-holomorphic sectional curvature* at x is $K(X,\phi X)$. If $K(X,\phi X)$ is a constant H for all points x in M and for all ϕ-holomorphic sections, then (M,Ω) is said to have *constant ϕ-holomorphic sectional curvature.*

Proposition 2.10 (Tanno, 1969). If (M,Ω) is a Sasakian DS and if $2n+1\geqslant 5$, then M always has constant ϕ-holomorphic sectional curvature, say H. And if $H>-3$, by a suitable choice of α, M has constant curvature 1 with respect to the deformed structure D^α.

Let S^{2n+1} be the unit sphere in Euclidean space E^{2n+2}. Let J be the natural complex structure on $\mathbf{C}E^{n+1}=E^{2n+2}$. Take $Z=Jx$ for unit vector x in S^{2n+1} and g the induced metric from E^{2n+2} onto S^{2n+1}. Then g and Z determine ω and ϕ by $\omega=g(Z,\cdot)$ and $d\omega(X,Y)=g(X,\phi Y)$; and (ϕ,Z,ω,g) is a Sasakian structure (*v.* Sasaki, 1965–1968; Tanno, 1969). Let D^α be as above with $\alpha = 4/(H+3) > 0$. Then D^α is a Sasakian structure on M with constant ϕ-holomorphic sectional curvature $H > -3$ (cf. Tanno, 1968); and denote S^{2n+1} with this structure by $S^{2n+1}[H]$. Let $E^{2n+1}[-3]$ be E^{2n+1} with the natural Sasakian structure and constant ϕ-holomorphic sectional curvature $H = -3$, as defined in Tanno (1969). Let $\mathbf{C}D^n$ be the open unit ball in $\mathbf{C}^n$, $L = \mathbf{R}$ and $(L, \mathbf{C}D^n)$ the product bundle $L \times \mathbf{C}D^n \to \mathbf{C}D^n$. There is a natural Sasakian structure on $(L, \mathbf{C}D^n)$ with constant ϕ-holomorphic sectional curvature $H < -3$, *v.* Tanno (1969). Denote this space by $(L, \mathbf{C}D^n)[H]$.

Proposition 2.11 (Tanno, 1969). If (M, Ω) is a connected and simply connected, complete Sasakian DS with constant ϕ-holomorphic sectional curvature H, then M is diffeomorphic to a homogeneous space $\mathrm{Aut}(M)/$(Isotropy group) and M is isomorphic (i.e. structurally preserving diffeomorphic) to:

(1) $S^{2n+1}[H]$ if $H > -3$; or M is D-homothetic to $S^{2n+1}[1]$;
(2) $E^{2n+1}[-3]$, if $H = -3$;
(3) $(L, \mathbf{C}D^n)[H]$, if $H < -3$.

These are all proper, regular (since they are homogeneous) cDS; and so they are of the form $G^1 \to M \to B$ where B is $\mathbf{C}P(n)$ in case (1), $\mathbf{C}^n$ in case (2), and $\mathbf{C}D^n$ in case (3) and $G^1 = \mathbf{R}$ or S^1.

Tanno has also classified connected Riemannian DSs with automorphism groups of maximal dimension. Namely,

Proposition 2.12 (Tanno, 1969). If (M, Ω) is a connected Riemannian DS of dimension $2n+1$, then $\dim \mathrm{Aut}(M) \leqslant (n+1)^2$. And the maximum is attained iff the sectional curvature for two-planes containing Z is a constant c and M is one of the following spaces:

(1) $c > 0$: a homogeneous Sasakian manifold (or its e-deformation) with constant ϕ-holomorphic sectional curvature H and:

(1a) $S^{2n+1}[H]/F(t_1)$ for $H > -3$ where $F(t_1)$ denotes a finite (cyclic) group generated by $\exp(t_1 Z)$ where $2\pi . 4(H+3)^{-1}/t_1$ is an integer;

(1b) $E^{2n+1}[-3]/F(t_2)$ for $H = -3$, where $F(t_2)$ is a cyclic group generated by $\exp(t_2 Z)$ where t_2 is a real number;

(1c) $(L, \mathbf{C}D^n)[H]/F(t_3)$ for $H < -3$ where t_3 is a real number;

(2) $c = 0$: six global Riemannian products

$$A \times \mathbf{C}P(n), \qquad A \pm \mathbf{C}E^n, \qquad A \times \mathbf{C}D^n$$

where $A = S^1$ or L;

(3) $c < 0$: a product space $Lx_{ct}\mathbf{C}E^n$ (*v.* Tanno, 1969).

Corollary 2.13. If (M, Ω) is a connected Sasakian DS then $\dim \mathrm{Aut}(M) = (n+1)^2$ iff (M, Ω) has constant ϕ-holomorphic sectional curvature and is one of (a), (b) or (c) in Proposition 2.12 above.

Corollary 2.14. If (M, Ω) is a compact, connected and simply connected Riemannian DS, in particular if (M, Ω) is a compact connected and simply connected QDS, and if $\dim \mathrm{Aut}(M) = (n+1)^2$, then M is a sphere with a Sasakian structure or its deformation; and by Hurt (1970) this QDS is the system of independent harmonic oscillators with equal periods.

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